Quantization of the Electromagnetic Field*

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A method of quantizing the electromagnetic field is proposed, where the underlying Hilbert space has a definite metric and the verifications of Lorentz covariance can be easily carried out with the help of Ward's identity. The usual Feynman-Dyson rules are obtained for the electron-photon system, and the longitudinal and scalar photons are found to have separately no interaction with the electron field.

1. INTRODUCTION

THE quantized form of electrodynamics has been known for over three decades.[1,2] One of the many problems encountered at a rather early stage in quantizing the electromagnetic field is that, if the Lorentz condition is considered to be a restriction on the state vectors, these state vectors cannot be normalized.[3] To avoid this difficulty, one of two rather different methods is generally used. In one method the longitudinal and scalar parts of the field are eliminated from the beginning and are thus not considered as dynamic variables,[2] while in the second method proposed by Gupta, Bleuler,[4] and others the indefinite metric of Dirac and Pauli[5] is employed. The first method has the feature that Lorentz covariance cannot be recognized in the intermediate steps and need be verified at the end, while a difficulty with the second method lies in the interpretation of negative probabilities.

Since both are quite complicated, it is proposed here to consider yet a third and possibly simpler method of avoiding this problem of unnormalizable state vectors. For a massless vector meson field, the four components A_μ are independent dynamic variables. In order to define the transverse, the longitudinal, and the scalar parts of the field, it is necessary to introduce polarization vectors. There is no unique way to define these polarization vectors; indeed, there is a four-dimensional family of possible polarization vectors for each momentum. A special choice was used by Fermi.[2] In the process of quantizing the field, because of the commutation rules, the creation and annihilation operators for the scalar field interchange their roles. For this reason, the bare vacuum depends on the choice of polarization vectors, and so does the bare propagator. Because of this arbitrariness, it is proposed in this paper to consider in a natural manner certain one-parameter families of normalizable state vectors. When the parameter increases without bound, the state vectors satisfy the Lorentz condition in the norm, although they do not approach a limit themselves. Using these vectors, the Feynman-Dyson rules are obtained for computing the S matrix, which is not Lorentz covariant and furthermore couples the transverse field to the longitudinal and scalar fields. In the limit, however, the usual Feynman-Dyson rules are obtained for transverse photons, while the longitudinal and scalar photons are found to have separately no interaction with the source field in the same limit and hence cannot be observed. In a later paper, it is hoped to apply similar considerations to the **b** field of Yang and Mills.[6]

Families of normalizable vectors have been used by Utiyama *et al.*[7] to study in detail a result of Dyson.[8]

As usual, unless otherwise stated the summation convention is used, with Greek indices running from 0 to 3 and Latin indices from 1 to 3. Units are chosen so that $c=\hbar=1$, and the metric is $(-1,+1,+1,+1)$ with $x^0=t$.

2. POLARIZATION VECTORS

Consider for the photon a given real four-momentum k_μ that is not zero and satisfies

$$k_\mu k^\mu=0. \tag{1}$$

For such a k_μ, the polarization vectors are four real vectors that satisfy

$$g_{\mu\nu}\epsilon_\sigma{}^\mu\epsilon_{\sigma'}{}^\nu=g_{\sigma\sigma'}, \tag{2}$$

and

$$\epsilon_\sigma{}^\mu k_\mu=0 \quad \text{for} \quad \sigma=1,2. \tag{3}$$

Here $\epsilon_\sigma{}^\mu$ is the μth component of the σth polarization vector. $\epsilon_1{}^\mu$ and $\epsilon_2{}^\mu$ are said to be transverse by (3), $\epsilon_3{}^\mu$ longitudinal and $\epsilon_0{}^\mu$ scalar. Since (2) and (3) give twelve conditions on the sixteen numbers $\epsilon_\sigma{}^\mu$, there is a four-dimensional family of possible polarization vectors.

Let $\mathcal{P}_0$ be the plane determined by the vectors ϵ_1 and ϵ_2, and let $\mathcal{P}$ be the plane orthogonal to $\mathcal{P}_0$. By (2), $\mathcal{P}_0$ is spacelike. Therefore, $\mathcal{P}$ contains two linearly independent lightlike vectors. Take one of them to be k_μ, and normalize the other one, called $\tilde{k}_\mu$, by

$$k^\mu\tilde{k}_\mu=1. \tag{4}$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A 114**, 243 (1927).

[2] E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).

[3] S. T. Ma, Phys. Rev. **75**, 535 (1950).

[4] S. N. Gupta, Proc. Phys. Soc. (London) **A 63**, 681 (1950); K. Bleuler, Helv. Phys. Acta. **23**, 567 (1950).

[5] P. A. M. Dirac, Proc. Roy. Soc. (London) **A 180**, 1 (1942); W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).

[6] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[7] R. Utiyama, T. Imamura, S. Sunakawa, and T. Dodo, Progr. Theoret. Phys. (Kyoto) **6**, 587 (1951).

[8] F. J. Dyson, Phys. Rev. **77**, 420 (1950).

By (2), the polarization vectors ϵ_0 and ϵ_3 are in $\mathcal{P}$. Since the over-all sign of each polarization vector is of no importance, ϵ_0 and ϵ_3 can be expressed as

$$\epsilon_0{}^\mu = (e^{\Phi}k^\mu - e^{-\Phi}\tilde{k}^\mu)/\sqrt{2},$$

and

$$\epsilon_3{}^\mu = (e^{\Phi}k^\mu + e^{-\Phi}\tilde{k}^\mu)/\sqrt{2}, \quad (5)$$

where Φ is real.

The four-dimensional family of possible polarization vectors has thus been parametrized by $\tilde{k}_\mu$ (two parameters) to fix $\mathcal{P}$ and $\mathcal{P}_0$, the direction of ϵ_1 in $\mathcal{P}_0$, and the number Φ. All these parameters are, in general, functions of k.

3. FREE ELECTROMAGNETIC FIELD

The free electromagnetic field may be described by the Lagrangian density

$$L = -\tfrac{1}{2}(\partial A_\mu/\partial x^\nu)(\partial A^\mu/\partial x_\nu), \quad (6)$$

where A_μ are the independent Hermitian dynamic variables. The variables conjugate to A_μ are

$$\pi^\mu = \partial L/\partial(\partial A_\mu/\partial t) = \partial A^\mu/\partial t. \quad (7)$$

Second quantization is easily carried out by Fourier series expansion in a cubical box of volume V

$$A_\mu(x) = V^{-1/2}\sum_{\mathbf{k}}(2|\mathbf{k}|)^{-1/2} \times[a_\mu(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i|\mathbf{k}|t} + a_\mu{}^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i|\mathbf{k}|t}], \quad (8a)$$

and

$$\pi^\mu(x) = -iV^{-1/2}g^{\mu\nu}\sum_{\mathbf{k}}(\tfrac{1}{2}|\mathbf{k}|)^{1/2} \times[a_\nu(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i|\mathbf{k}|t} - a_\nu{}^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i|\mathbf{k}|t}], \quad (8b)$$

together with the commutation rule

$$[a_\mu(\mathbf{k}), a_{\mu'}{}^*(\mathbf{k}')] = g_{\mu\mu'}\delta_{\mathbf{k}\mathbf{k}'}. \quad (9)$$

The operators $c_\sigma(\mathbf{k})$ for the annihilation of scalar, transverse, or longitudinal photons are related to $a_\mu(\mathbf{k})$ by

$$a_\mu(\mathbf{k}) = g_{\mu\nu}g^{\sigma\tau}\epsilon_\sigma{}^\nu(\mathbf{k},\Phi_{\mathbf{k}})c_\tau(\mathbf{k},\Phi_{\mathbf{k}}). \quad (10)$$

The polarization vector, and hence the operators c_σ, depends explicitly not only on $k=(|\mathbf{k}|,\mathbf{k})$, but also on $\tilde{k}(\mathbf{k})$ and $\Phi_{\mathbf{k}}$. In (10), only the dependence on $\mathbf{k}$ and $\Phi_{\mathbf{k}}$ are written explicitly. The commutation rule for c_σ follows immediately from (9)

$$[c_\sigma(\mathbf{k},\Phi_{\mathbf{k}}), c_{\sigma'}{}^*(\mathbf{k}',\Phi_{\mathbf{k}'})] = g_{\sigma\sigma'}\delta_{\mathbf{k}\mathbf{k}'}. \quad (11)$$

Given a choice of polarization vectors for each $\mathbf{k}$, the bare vacuum is defined by

$$c_0{}^*(\mathbf{k},\Phi_{\mathbf{k}})|\tilde{k},\Phi\rangle = 0 \quad (12a)$$

and

$$c_j(\mathbf{k},\Phi_{\mathbf{k}})|\tilde{k},\Phi\rangle = 0. \quad (12b)$$

If the number operators are defined as usual by

$$N_0(\mathbf{k},\Phi_{\mathbf{k}}) = c_0(\mathbf{k},\Phi_{\mathbf{k}})c_0{}^*(\mathbf{k},\Phi_{\mathbf{k}}) \quad (13a)$$

and

$$N_j(\mathbf{k},\Phi_{\mathbf{k}}) = c_j{}^*(\mathbf{k},\Phi_k)c_j(\mathbf{k},\Phi_k), \quad \text{(no sum)}, \quad (13b)$$

then the total energy and momentum are given by

$$H = \sum_{\mathbf{k}}|\mathbf{k}|[-N_0(\mathbf{k},\Phi_{\mathbf{k}}) + N_1(\mathbf{k}) + N_2(\mathbf{k}) + N_3(\mathbf{k},\Phi_{\mathbf{k}})], \quad (14a)$$

and

$$\mathbf{P} = \sum_{\mathbf{k}}\mathbf{k}[-N_0(\mathbf{k},\Phi_{\mathbf{k}}) + N_1(\mathbf{k}) + N_2(\mathbf{k}) + N_3(\mathbf{k},\Phi_{\mathbf{k}})]. \quad (14b)$$

Note that the energy is not positive definite, and the bare vacuum is degenerate with many other states. It is for this reason that the bare vacuum depends on the choice of polarization vectors. In particular,

$$|\tilde{k},\Phi\rangle = \prod_{\mathbf{k}} \operatorname{sech}\Phi_{\mathbf{k}}\, e^{-\tanh\Phi_{\mathbf{k}} c_3{}^*(\mathbf{k},0)c_0(\mathbf{k},0)}|\tilde{k},0\rangle. \quad (15)$$

3. THE LIMIT $\Phi\rightarrow\infty$

Because of the ambiguity in the bare vacuum with respect to photons, some additional rule must be employed to obtain definitive results for electrodynamics.

The bare vacuum $|k,\Phi\rangle$ may be used to compute the bare photon propagator with the result that

$$\langle\tilde{k},\Phi|T[A_\mu(x)A_\nu(x')]|\tilde{k},\Phi\rangle = -ig_{\mu\nu}\Delta_F(x-x') + (2V)^{-1}\sum_{\mathbf{k}}(2|\mathbf{k}|)^{-1} \times[e^{2\Phi_{\mathbf{k}}}k_\mu k_\nu - (k_\mu\tilde{k}_\nu + k_\nu\tilde{k}_\mu) + e^{-2\Phi_{\mathbf{k}}}\tilde{k}_\mu\tilde{k}_\nu] \times[e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i|\mathbf{k}|(t-t')} + e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-i|\mathbf{k}|(t'-t)}]. \quad (16)$$

Consequently, the Fourier transform of the bare propagator is given by

$$D_{\mu\nu}(k;\tilde{k},\Phi) = -ig_{\mu\nu}(k^2-i\epsilon)^{-1} + 2\pi\delta(k^2) \times[e^{2\Phi_{\mathbf{k}}}k_\mu k_\nu - (k_\mu\tilde{k}_\nu + k_\nu\tilde{k}_\mu) + e^{-2\Phi_{\mathbf{k}}}\tilde{k}_\mu\tilde{k}_\nu]. \quad (17)$$

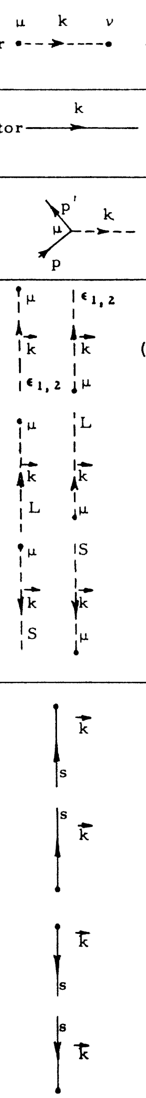
Since there is no way to specify naturally how $\tilde{k}$ should be chosen, a meaningful theory must be independent of this choice. Therefore, the form of (17) indicates that the physical world can only be related to the limit

$$\Phi_{\mathbf{k}} \rightarrow \infty. \quad (18)$$

Since it follows from (15) that $\lim_{\Phi\rightarrow\infty}|\tilde{k},\Phi\rangle$ does not exist, results of quantum electrodynamics are to be obtained by the following *rule*: In order to obtain an element of the S matrix in quantum electrodynamics, it is computed by first using $|\tilde{k},\Phi\rangle$ as the bare vacuum, and then taking the limit (18).

4. FEYNMAN-DYSON RULES

Using a bare vacuum specified by $\tilde{k}$ and Φ, The Feynman-Dyson rules for the electron-photon system are those given in Fig. 1. The notations used there are as follows: n is the number of vertices, N_e is the number of external lines, N_i is the number of internal lines, and N_l is the number of electron loops in the Feynman-Dyson graph under consideration. In the first column of Fig. 1, the usual Feynman-Dyson rules are shown; quantities that approach zero in the limit (18) are given in the third column; and the rest in the second column.



Photon Propagator	$-ig_{\mu\nu}\dfrac{1}{k^2-i\epsilon}$	$2\pi\,\delta(k^2)\{e^{2\Phi\vec{k}}k_\mu k_\nu-(k_\mu\tilde{k}_\nu+k_\nu\tilde{k}_\mu)\}$	$2\pi\,\delta(k^2)e^{-2\Phi\vec{k}}\tilde{k}_\mu\tilde{k}_\nu$
Electron Propagator	$i\dfrac{i\gamma^\mu k_\mu-m}{k^2+m^2-i\epsilon}$		
Vertex	$-e\gamma^\mu\delta^4(p-p'-k)$		
External Lines: Transverse Photon	$(2\lvert\vec{k}\rvert)^{-\frac{1}{2}}\epsilon_{1,2\mu}(\vec{k})$		
External Lines: Longitudinal Photon		$\frac{1}{2}\lvert\vec{k}\rvert^{-\frac{1}{2}}e^{\Phi\vec{k}}k_\mu$	$\frac{1}{2}\lvert\vec{k}\rvert^{-\frac{1}{2}}e^{-\Phi\vec{k}}\tilde{k}_\mu$
External Lines: Scalar Photon		$-\frac{1}{2}\lvert\vec{k}\rvert^{-\frac{1}{2}}e^{\Phi\vec{k}}k_\mu$	$\frac{1}{2}\lvert\vec{k}\rvert^{-\frac{1}{2}}e^{-\Phi\vec{k}}\tilde{k}_\mu$
External Lines: Electron ($s=\pm1$)	$u(\vec{k},s)$		
	$\bar{u}(\vec{k},s)$		
External Lines: Positron ($s=\pm1$)	$\bar{v}(-\vec{k},s)$		
	$v(-\vec{k},s)$		
Overall Factor of 2π:	$(2\pi)^{4n-\frac{3}{2}N_e-4N_i}$		
Overall Sign	$(-1)^{N_l}$		

FIG. 1. Feynman-Dyson rules in momentum space.

It remains to study the limit (18). Dyson[8] has pointed out that in $D_{\mu\nu}$ a term proportional to either k_μ or k_ν cannot contribute to the S matrix if the electrons are on their mass shells. In perturbation theory this follows from Ward's identity[9]

$$\frac{ik-m}{k^2+m^2-i\epsilon}-\frac{ik'-m}{k'^2+m^2-i\epsilon}=\frac{ik-m}{k^2+m^2-i\epsilon}i(k-k')\frac{ik'-m}{k'^2+m^2-i\epsilon}. \quad (19)$$

[9] J. C. Ward, Proc. Phys. Soc. (London) **A 64**, 54 (1951).

Therefore, in the computation of an element of the S matrix on mass shell, the terms appearing in the second column of Fig. 1 do not contribute and hence may be omitted. After this omission, those in the third column are seen to contribute nothing in the limit (18). Therefore, the usual Feynman-Dyson rules are obtained together with the statement that the longitudinal and scalar photons separately do not interact with the electron field. In other words, the S matrix consists of diagonal blocks so that the numbers of longitudinal and scalar photons of each momentum are separately conserved. The block with no longitudinal or scalar photons corresponds to physical reality.

5. DISCUSSION

One of the major problems with a quantization procedure where a definite metric is used and where A_μ are considered to be four independent fields is the following.[10] The commutator $[\partial A_\mu(x)/\partial x_\mu, A_\nu(x')]$ is a c number and does not vanish. How is this to be reconciled with the observation that its vacuum expectation value seems to be zero because of the Lorentz condition on the state vectors? Within the present formulation, the answer is as follows: It has been shown that every element of the S matrix on the mass shell formally has a limit under (18). But quantities that are not gauge invariant often have matrix elements that do not approach a limit under (18). For example, it is readily verified that

$$\langle \tilde{k},\Phi | c_3(\mathbf{k},\Phi_\mathbf{k}) A_\mu(x) | \tilde{k},\Phi\rangle = \tfrac{1}{2} V^{-1/2} |\mathbf{k}|^{-1/2} (e^{\Phi_\mathbf{k}} k_\mu + e^{-\Phi_\mathbf{k}} \tilde{k}_\mu) e^{-i\mathbf{k}\cdot\mathbf{x}+i|\mathbf{k}|t}. \quad (20)$$

Even though $\partial A_\mu/\partial x_\mu$ operated on a physical state does lead to a small factor of the form $e^{-\Phi}$, this small factor is cancelled by a corresponding large factor from A_ν as seen from (20). Thus, the vacuum expectation value of the above commutator is not zero.

The Lorentz condition on the state vector has not been explicitly stated so far. It follows from (15) that

$$[c_0(\mathbf{k},0)+c_3(\mathbf{k},0)] | \tilde{k},\Phi\rangle = (1-\tanh\Phi_\mathbf{k}) c_0(\mathbf{k},0) | \tilde{k},\Phi\rangle, \quad (21)$$

and

$$[c_0^*(\mathbf{k},0)+c_3^*(\mathbf{k},0)] | \tilde{k},\Phi\rangle = (1-\tanh\Phi_\mathbf{k}) c_3^*(\mathbf{k},0) | \tilde{k},\Phi\rangle. \quad (22)$$

Consequently, in the limit (18), the Lorentz condition is satisfied in norm. Thus, the Lorentz condition is naturally satisfied if any sense is to be made of describing the massless vector meson field by four independent fields without indefinite metric. This situation is quite different from the case of the massive vector meson field treated by Lee and Yang.[11]

Finally, it is to be noted that in the limit (18) each $\Phi_\mathbf{k}$ approaches infinity independently, although some uniformity is probably desired. Since the derivation of the usual Feynman-Dyson rules under this wide class of limiting processes depends critically on the validity of Ward's identity, the corresponding situation with other gauge fields can be much more complicated.

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[10] This point was first raised by Professor Yang. See S. T. Ma, Phys. Rev. **80**, 729 (1950).

[11] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).